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hyperbola. The points

$$\left( \pm a \left( \frac{\sqrt{a^2 + b^2}}{a} - 1 \right), 0 \right)$$

are called foci, . . . . The ratio of the distance from a point on the curve from

$$\left( a \left( \frac{\sqrt{a^2 + b^2}}{a} - 1 \right), 0 \right)$$

to its distance from the line

$$x^2 = \frac{a^2}{a^2 + b^2}$$

has the constant value

$$\frac{\sqrt{a^2 + b^2}}{a} \dots$$

The lines

$$x = \pm \frac{a^2}{a^2 + b^2}$$

are known as directrices.

All in twenty-four lines.

To avoid the charge of partisanship let me add that a glance at the *eleventh* edition of the *Encyclopædia Britannica* reveals, in one short paragraph, the near facts:<sup>1</sup>

Analytically the hyperbola is given by  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$  wherein  $ab > h^2$ . . . . In the rectangular hyperbola  $a = b$ ; hence its equation is  $x^2 - y^2 = 0$  . . . .

How can we believe and teach the doctrine that (as Bacon might have said it) "Mathematicks maketh the careful man" if such inaccurate statements are permitted in books which the public has a right to regard as sources of exact information?

### III. GEOMETRIC PROOFS OF THE LAW OF TANGENTS.

By H. C. BRADLEY, Massachusetts Institute of Technology.

Let  $ABC$  be the given triangle,  $\angle A$  acute and greater than  $\angle B$ . Produce  $AC$ . Lay off  $\angle ABD = \angle A$  on the same side of  $AB$  as  $C$ , forming the isosceles triangle  $ABD$ . Let  $E$  be the middle point of  $AB$ . Draw  $DE$ .

Produce  $BC$ . Lay off  $CF = AC$ . Draw  $AF$ , thus forming the isosceles triangle  $ACF$ . Let  $G$  be the middle point of  $AF$ . Draw  $GC$ , and produce it to meet  $DE$  at  $O$ .

Connect  $O$  with  $B$ . From  $O$  draw  $OH$  perpendicular to  $BC$ . Then

$$OH = BH \tan \angle OBH = CH \tan \angle OCH. \quad (1)$$

In the triangle  $BCD$ ,  $CO$  and  $DO$  bisect the angles at  $C$  and  $D$  respectively. Hence  $BO$  bisects the angle at  $B$ . Whence  $\angle OBH = \frac{1}{2}(A - B)$ . Also, we have  $\angle OCH = \frac{1}{2}(A + B)$ .

In the triangle  $ABF$ ,  $OE$  and  $OG$  are, respectively, perpendicular bisectors of the sides  $AB$  and  $AF$ . Hence  $OH$  is the perpendicular bisector of  $BF$ , and  $BH = HF$ . Whence  $BH = \frac{1}{2}(a + b)$ , and  $CH = \frac{1}{2}(a - b)$ .

<sup>1</sup> Vol. 14, p. 199.

Substituting in (1),  $\frac{1}{2}(a+b) \tan \frac{1}{2}(A-B) = \frac{1}{2}(a-b) \tan \frac{1}{2}(A+B)$ , which is the law of tangents.<sup>1</sup>

To complete the proof, we should also consider the cases in which  $\angle A$  is obtuse, and a right angle. These cases can safely be "left to the student." If  $\angle A$  is obtuse, the point  $O$  is no longer an in-center of the triangle  $CBD$ , but is one of its out-centers, while if  $\angle A = 90^\circ$  the triangle  $CBD$  reduces to two parallel lines. But the triangle  $OBC$ , in which the required relation appears so simply, can always be found.

The construction fails when the angles  $A$  and  $B$  are equal, but then the law itself is trivial.

By T. YAMANOUTI, Sixth National College, Okayama, Japan.

Let  $a > b$ . Draw  $CD$  the bisector of the angle  $C$ , meeting  $AB$  at  $D$ . Draw  $AM$  and  $BN$  perpendicular to  $CD$ , meeting  $CD$  and its extension at  $M$  and  $N$  respectively. Then from the similarity of the triangles  $AMC$ ,  $BNC$ ,

$$\frac{a}{b} = \frac{CN}{CM},$$

and

$$\frac{a+b}{a-b} = \frac{CN+CM}{CN-CM} = \frac{CN+CM}{DN+DM} = \frac{BN \tan NBC + AM \tan MAC}{BN \tan NBD + AM \tan MAD};$$

but

$$\angle MAC = \angle NBC = \frac{A+B}{2}; \quad \angle MAD = \angle NBD = \frac{A-B}{2};$$

therefore

$$\frac{a+b}{a-b} = \frac{\tan \frac{A+B}{2}}{\tan \frac{A-B}{2}}.$$

By W. V. LOVITT, Colorado College.

In this MONTHLY, 1920, 465, I gave six new proofs of the law of tangents. In this paper are given further proofs which are believed to be new. In the first proof a well-known formula is derived. By specializing the values of the variables therein the law of tangents is derived. Thus the law of tangents appears as a special case of a more general theorem. The general theorem is useful in solving problems in mechanics relating to three forces in equilibrium.<sup>2</sup> In view of the usefulness of the general theorem in mechanics it would seem desirable to have it included in elementary trigonometry.

In the triangle  $ABC$  let  $D$  be any point of  $BC$ ; call the parts into which  $AD$  divides the angle  $A$ ,  $\mu = BAD$ ,  $\nu = DAC$ ; call  $\theta = CDA$ . Let  $BE$  and  $CF$

<sup>1</sup> The same construction on a sphere will give one of Napier's analogies.

<sup>2</sup> Consult Miller and Lilly, *Analytic Mechanics*, New York, 1915, pp. 78, 108, 111.

be perpendicular to  $AD$ . We may suppose  $\theta$  acute. Then

$$\begin{aligned} BC \cos \theta &= BD \cos \theta + DC \cos \theta \\ &= DE + DF = AE - AF \\ &= BE \cot \mu - CF \cot \nu \\ &= BD \sin \theta \cot \mu - CD \sin \theta \cot \nu. \end{aligned}$$

That is,

$$BC \cot \theta = BD \cot \mu - CD \cot \nu.$$

To prove the law of tangents, take  $CD = b$ , where  $b < a$ . Then  $BD = a - b$ ,  $\mu = \frac{1}{2}(A - B)$ ,  $\theta = \nu = \frac{1}{2}(A + B)$ ; so that

$$a \cot \frac{1}{2}(A + B) = (a - b) \cot \frac{1}{2}(A - B) - b \cot \frac{1}{2}(A + B).$$

Whence

$$\frac{a + b}{a - b} = \frac{\tan \frac{1}{2}(A + B)}{\tan \frac{1}{2}(A - B)}.$$

We may give the proof also by use of the perpendicular  $BE$  alone instead of both  $CF$  and  $BE$ . Project the sides  $a, b, c$  on  $BE$  and  $AD$  and we find

$$c \sin \frac{1}{2}(A - B) = (a - b) \sin \frac{1}{2}(A + B), \quad c \cos \frac{1}{2}(A - B) = (a + b) \cos \frac{1}{2}(A + B).$$

Dividing,

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B).$$

If we use  $C/2$  instead of  $\frac{1}{2}(A + B)$ , we have

$$c \sin \frac{1}{2}(A - B) = (a - b) \cos \frac{C}{2}, \quad c \cos \frac{1}{2}(A - B) = (a + b) \sin \frac{C}{2}.$$

These are the Mollweide equations. Dividing,

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \frac{C}{2}.$$

In view of the usefulness of the Mollweide equations in checking a solution by the law of tangents and the ease, as here shown, in deriving them, it would seem that their more general inclusion in elementary trigonometry texts is desirable.

#### HISTORICAL AND BIBLIOGRAPHICAL NOTES.

By R. C. ARCHIBALD, Brown University.

The formulæ

$(b + c) \sin \frac{1}{2}A = a \cos \frac{1}{2}(B - C)$ , and  $(b - c) \cos \frac{1}{2}A = a \sin \frac{1}{2}(B - C)$ , were first given in T. Simpson, *Trigonometry Plane and Spherical*, London, 1748, pages 60-61. If instead of  $\sin \frac{1}{2}A$  we substitute  $\cos \frac{1}{2}(B + C)$ , and instead of  $\cos \frac{1}{2}A$ ,  $\sin \frac{1}{2}(B + C)$ , we have, in effect, formulæ given in [F. W. v. Oppell, *Analysis Triangulorum*, 1746, page 18. Sir Isaac Newton gave, in effect, yet another form to the first of these formulæ,<sup>1</sup> in *Arithmetica Universalis*, Cam-

<sup>1</sup> This was in his discussion of the familiar problem: To determine the sides of a triangle, given the base  $AB$ , the sum of the sides  $AC + BC$ , and the vertical angle  $C$ .

bridge, 1707, page 122, where  $\sin AEC$  is substituted for  $\cos \frac{1}{2}(B - C)$  [ $E$  being the point, corresponding to  $D$  in Professor Lovitt's discussion, when  $AD$  is drawn bisecting the angle  $A$ .]

Simpson's formulæ were given by Mollweide, without reference to Simpson, in Zach's *Monatliche Correspondenz*, Gotha, volume 18, 1808, page 396.

To various geometrical proofs of the Law of Tangents already indicated in this MONTHLY (1920, 53-54, 465-467; 1921, 71, 79, 170-171), might be added: one by Vignal in *Nouvelles Annales de Mathématiques*, volume 3, 1844, pages 456-457; and one by John Keill, the earliest I have met with, given in his anonymously published *Trigonometriæ Planæ & Sphæricæ*, Oxford, 1715, pp. 16-17.

#### IV. SOME FORMULAS OF ELEMENTARY TRIGONOMETRY.

By W. J. RUSK, Grinnell College.

The formulas that are taken for granted are the sine formulas:

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R},$$

and the projection formulas:

$$a = b \cos \gamma + c \cos \beta, \quad b = c \cos \alpha + a \cos \gamma, \quad c = a \cos \beta + b \cos \alpha.$$

If we multiply these in order by  $a$ ,  $b$ ,  $c$ , and then add the first two results and subtract the third we get one of the cosine formulas; so we shall consider them as given also.

Consider the triangle  $ABC$  with  $b < a$ ; take  $D$  on  $AB$  so that  $CA = CD$ ; then  $\angle DCB = \alpha - \beta$  and

$$DB = a \cos \beta - b \cos \alpha = \frac{a^2 - b^2}{c}.$$

1. Formulas for  $\sin (\alpha + \beta)$  and  $\sin (\alpha - \beta)$ . We have from triangle  $ABC$ ,

$$\frac{\sin \gamma}{c} = \frac{\sin (\alpha + \beta)}{a \cos \beta + b \cos \alpha} = \frac{1}{2R};$$

$$\begin{aligned} \therefore \sin (\alpha + \beta) &= \frac{a}{2R} \cos \beta + \frac{b}{2R} \cos \alpha \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta. \end{aligned}$$

and from triangle  $CDB$ ,

$$\frac{\sin \beta}{b} = \frac{\sin (\alpha - \beta)}{a \cos \beta - b \cos \alpha} = \frac{1}{2R},$$

or

$$\begin{aligned} \sin (\alpha - \beta) &= \frac{a}{2R} \cos \beta - \frac{b}{2R} \cos \alpha, \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta. \end{aligned}$$